Name:____________________ M#:__________ Instructor:____________________

Instructions: This exam has seven problems on seven pages. Show all your work, expressing yourself in clear and concise manner. Do as many problems as you can, but be advised that a complete solution to a problem may be worth more than several partial ones. Use the backs of the exam pages for work, if necessary. No calculators of any kind are allowed.

1. Calculate $\displaystyle\int_0^{\pi/2} \sqrt{1+\sin x}\, dx$.

   Hint: Use $1 = \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}$. (Or do it your way.)

Solution:

Also use $\sin(2(x/2)) = 2\sin(x/2)\cos(x/2)$ two write the integrand as the square root of the square of a positive number.

Obtain $\displaystyle\int_0^{\pi/2} \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\, dx = 2.$

2. Integrate.

$$\int \frac{x\arctan x}{(1+x^2)^2}\,dx$$

Solution: We have

$$\begin{aligned}
\int \frac{x\arctan x}{(1+x^2)^2}\,dx &= (\textit{parts}) = -\frac{1}{2}\,\frac{1}{1+x^2}\arctan x + \frac{1}{2}\int \frac{dx}{(1+x^2)^2} \\
&= (x = \tan\theta) = -\frac{1}{2}\,\frac{1}{1+x^2}\arctan x + \frac{1}{2}\int \frac{d\theta}{\sec^2\theta} \\
&= -\frac{1}{2}\,\frac{1}{1+x^2}\arctan x + \frac{1}{2}\int \cos^2\theta\,d\theta \\
&= -\frac{1}{2}\,\frac{1}{1+x^2}\arctan x + \frac{1}{4}\int (1+\cos(2\theta))\,d\theta \\
&= -\frac{1}{2}\,\frac{1}{1+x^2}\arctan x + \frac{1}{4}(\theta + \frac{1}{2}\sin(2\theta)) + C \\
&= -\frac{1}{2}\,\frac{1}{1+x^2}\arctan x + \frac{1}{4}\,\arctan x + \frac{1}{4}\,\frac{x}{1+x^2} + C \\
&= \boxed{\frac{x + (x^2-1)\arctan x}{4(1+x^2)} + C}
\end{aligned}$$

3. Suppose you're standing on top of a light house so your eyes are 100 feet above the level of a calm sea. You look out to where the sea and sky meet. About how far is that horizon? Give your answer in miles and provide some estimate of your accuracy. You can assume that the Earth is a sphere with radius 4000 miles. There 5280 feet in a mile.

Solution: Let R be Earth's radius, T the height of the eye above the sea, and D the straight line distance to the horizon. Then, since tangents to circles are perpendicular to radii, Pythagoras assures us that

$$(R+T)^2 = D^2 + R^2$$

So that

$$D = \sqrt{2RT + T^2} = \sqrt{2RT}\sqrt{1 + \frac{1}{2}\frac{T}{R}}.$$

This formula is, of course, exact. The approximation part comes in when we try to evaluate it as a decimal number.

The ratio T/R has magnitude 10^{-4} so dropping the last factor won't alter the first few decimal places in our answer. (That is, linear approximation tells us $\sqrt{1 + 10^{-4}} \approx 1 + (1/2)10^{-4}$.)

In square miles

$$\begin{aligned} RT &\approx 4 \times 10^3 \times 10^2/(5.3 \times 10^3) \\ &\approx (4/5.3) \times 10^2 \end{aligned}$$

So

$$D \approx \sqrt{(8/5.3)10^2} \approx 1.2 \times 10\text{miles}$$

and we're confident that the two digits given are correct. We can say that the horizon is a bit more than 12 miles from the top of a 100 foot tower.

4. For any positive integer n
$$n^2 = n + n + \cdots + n\,,$$
where the sum on the right has n terms. Differentiating both sides with respect to n we obtain
$$2n = 1 + 1 + \cdots + 1\,,$$
or
$$2n = n\,.$$
Dividing by n,
$$2 = 1\,.$$
Is there anything wrong with this argument? Explain.

Solution: There's nothing wrong with the argument, the conclusion is perfectly correct.

More seriously, yes, there is something wrong because the argument starts with a correct formula and ends by concluding $2 = 1$, which isn't true.

If the formula we start with is true for integers $n \geq 1$, why can we differentiate it with respect to n? We can't, if you think about the definition of derivative.

Other people argued that the variable n in the formual appears in *two* places: first "$n^2 = n + n + \cdots + n$" and second in "where the sum on the right has n terms." We don't have a method for differentiating in such settings and so the "derivative" in the argument isn't correct.

5. At what points on the graph of $y = x^2 e^{-x}$ does the tangent line have y-intercept equal to 0?

Solution: At $x = 0, 1$.

The tangent line at $(a, f(a))$ passes through $(0, 0)$ exactly when $f(a)/a = f'(a)$. This equation, which amounts to $(2 - a)ae^{-a} = ae^{-a}$, has solutions $a = 0$ and $a = 1$.

6. A particle is moving in the xy-plane along the polar curve $r = 2\sin(3\theta)$, $0 \leq \theta \leq \pi$. Its polar angle, θ, is a function of time, t. When the particle is at the point $(0, -1)$, θ is changing at a rate $\frac{d\theta}{dt} = -3$. How fast is the x-coordinate of the particle changing at that point?

Solution: We have $x = r\cos(\theta) = 2\sin(3\theta)\cos(\theta)$. Therefore,

$$\frac{dx}{dt} = \frac{d}{d\theta}\left[2\sin(3\theta)\cos(\theta)\right]\frac{d\theta}{dt} = 2\big[3\cos(3\theta)\cos(\theta) - \sin(3\theta)\sin(\theta)\big]\frac{d\theta}{dt}$$

The only value θ in the interval $[0, \pi]$ that corresponds to the Cartesian point $(0, -1)$ is $\theta = \frac{\pi}{2}$. Hence,

$$\left.\frac{dx}{dt}\right|_{(x,y)=(0,-1)} = 2\big[3\cos(3\theta)\cos(\theta) - \sin(3\theta)\sin(\theta)\big]_{\theta=\pi/2}\big(-3\big) = \boxed{-6}$$

7. Let g be a continuous function on the interval $[0, 1]$. Assume that $\int_0^1 g(x)\,dx = 0$. Show that
$$\int_0^1 e^{g(x)} g(x)\,dx \geq 0.$$

Solution: For any x we have
$$e^{g(x)} g(x) \geq g(x). \tag{*}$$

Indeed, if $g(x) > 0$, this inequality is equivalent to the true inequality $e^{g(x)} \geq 1$. If $g(x) < 0$, this inequality is equivalent to the true inequality $e^{g(x)} \leq 1$. Finally, if $g(x) = 0$, this holds with equality.

Now, (*) implies that
$$\int_0^1 e^{g(x)} g(x)\,dx \geq \int_0^1 g(x)\,dx = 0 \qquad \square$$